4.3: Linearly independent sets & Bases.

Key idea: Defining linear independence for any vector space allows us to consider spanning sets for vector spaces and subspaces of minimal size. We call such sets bases and use them to efficiently describe and work with a space.

We have already considered spanning sets for vector spaces, we now wish to discuss minimal spanning sets i.e. ones with no extraneous members. Eventually we will discuss other optimal spanning sets. Toward these ends we extend linear indepence to arbitrary vector spaces:

Def: An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent if the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution $c_1 = c_2 = \dots = c_p = 0$.

The set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent if there is a nontrivial solution. In this case the above equation becomes a linear dependence relation.

As before, a set of vectors is linearly dependent if one of of the vectors is in the span of the others. Notice if V is not $\mathbb{R}^n$, solving $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$ cannot be accomplished by using a system of linear equations $A\vec{x} = \vec{0}$. Instead, we must argue using linear combinations and span.

Fact: $\{\vec{v}_1, \dots, \vec{v}_p\}$ (with $\vec{v}_1 \neq \vec{0}$) is linearly dependent if and only if some $\vec{v}_j$ is a linear combination of the previous vectors $\vec{v}_1, \dots, \vec{v}_j$.

Ex In $\mathbb{P}$ (the vector space of all polynomials) the set $\{p_1, p_2, p_3\}$ with

$p_1(t) = t^2$ is linearly dependent because p_3 in Span$\{p_1, p_2\}$

$p_2(t) = t + 2$

$p_3(t) = 4t^2 - t - 2$ $\qquad p_3 = 4t^2 - t - 2 = 4p_1 - p_2$.

Ex In the space of continuous functions $C[0,1]$, $\{\sin t, \cos t\}$ is and $\{\sin t \cdot \cos t, \sin 2t\}$ is not linearly independent. $\quad 2\sin t\cos t = \sin 2t$

Recall that a spanning set generates a subspace. We now consider linearly independent spanning sets which we call **bases**.

Def: Let H be a subspace of V. Then a set of vectors $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a **basis for H** if

1) $\mathcal{B}$ is linearly independent
2) $\mathcal{B}$ is a spanning set for H: $H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$

So a basis is a set of vectors exactly large enough to describe all other vectors in the space (as a linear combination).

Ex The columns of $I_n = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$ form a basis for $\mathbb{R}^n$.

Clearly

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

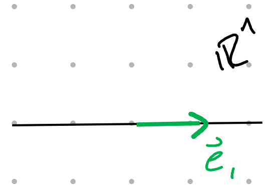
are linearly independent. ($c_1\vec{e}_1 + \cdots + c_n\vec{e}_n = \vec{0}$ only if $c_i = 0$ for all i).

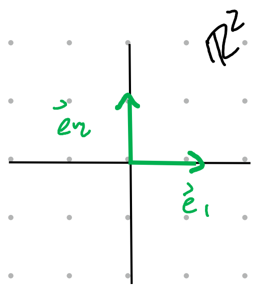
And $\text{Span}\{\vec{e}_1, \dots, \vec{e}_n\} = \mathbb{R}^n$ as for any $\vec{x}$ in $\mathbb{R}^n$ we have

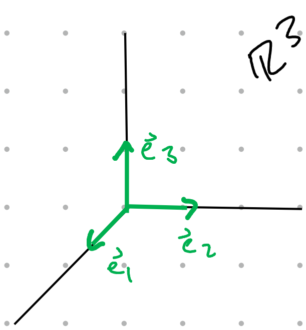
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

This basis is so intuitive and ubiquitous we refer to $\{\vec{e}_1, \dots, \vec{e}_n\}$ as **the standard basis for $\mathbb{R}^n$**.

Geometrically we think of as a basis as all that is needed to "get everywhere" in a vector space by moving through the span.







Ex The **standard basis of $\mathbb{P}_n$** is

$S = \{1, t, t^2, \dots, t^n\}$. Why?

1) Clearly $\text{Span}\, S = \mathbb{P}_n$

2) $\vec{0}$ in $\mathbb{P}_n = 0 + 0t + \cdots + 0t^n \Rightarrow S$ is lin. independent

Ex| Consider $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$?

Note:

$$A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$ is invertible (Row reduce or find $\det(A)$.)

So by the IMT,

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent and $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

So yes, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a (nonstandard) basis for $\mathbb{R}^3$.

Note this implies the columns of <u>any</u> invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

We next turn to recover bases for a subspace from any spanning set. It follows that any basis is a spanning set for a subspace H but not vice versa.

Ex| $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \right\}$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right\}$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix} \right\}$

linearly independent but does not span $\mathbb{R}^3$ | linearly independent and spanning set for $\mathbb{R}^3$ | spanning set for $\mathbb{R}^3$ but not linearly independent

Basis? ✗ ✓ ✗

Fact: Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a spanning set for a subspace H of a v.space V. Then some subset of S is a basis for H.

Why? Just remove all linearly dependent elements until you have a lin. independent spanning set.

We can use this fact to find a basis for the column space of an $m \times n$ matrix A.

How? Use only the linearly independent (pivot!) columns of A.

Ex) Let $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 2 \\ 3 & 1 & 5 & 1 & 4 \\ 2 & -1 & 0 & 1 & 2 \\ 5 & 0 & 5 & 0 & 0 \end{bmatrix}$. Find a basis for Col(A).

Note $\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix} \right\}$ and

Col(A) is a subspace of $\mathbb{R}^4$, so a basis can't have more than 4 vectors and still be linearly independent. To determine which columns of A are linearly independent we row reduce:

$A \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which implies that $\vec{a}_1, \vec{a}_2, \vec{a}_4$ are pivot columns and $\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$ and $\vec{a}_5 = \vec{a}_2 + 3\vec{a}_4$.

So $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$ are linearly independent and $\vec{a}_3, \vec{a}_5$ in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$

Thus a basis for Col(A) is

$$\mathcal{B} = \{\vec{a}_1, \vec{a}_2, \vec{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

pivot columns of A, not of reduced echelon form.

Note: $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}$ in Col(A) but not in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

In general, the pivot columns of A always form a basis for Col(A).

How do we find a basis for Null(A)? Follow the procedure of 4.2. The spanning set we found in those examples is automatically linearly independent and therefore a basis.